

The quenched central limit theorem for a model of random walk in random environment

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March 22, 2017

Abstract

A short proof of the quenched central limit theorem for the random walk in random environment introduced by Boldrighini, Minlos, and Pellegrinotti [BMP94] is given.

1 Introduction

In this article we give a short proof of the quenched central limit theorem for a model of the random walk in random environment. At each site the transition probability kernel is affected by the current state of the environment at this site. The model was introduced by Boldrighini, Minlos, and Pellegrinotti, see in particular [BMP94, BMP97, BMP07] and more recent papers Boldrighini et al. [BMPZ15] and Di Persio [DP10]. The model is described in Section 2. Boldrighini et al. [BMPZ15] contains a nice overview of the literature on the subject. For a survey on the recent progress on this and similar models see Zeitouni [Zei06] or Biskup [Bis11]. A related model is considered by Barraquand and Corwin [BC16] and Thiery and Le Doussal [TLD17].

The proof makes use of the multidimensional martingale CLT by Küchler and Sørensen [KS99]. The paper is organized as follows. In Section 2 we describe the model and give the statement. In Section 3 we give the proofs and some further comments.

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2 Model, conditions and results

Consider a particle moving in a n -dimensional infinite lattice and denote by X_t is position at time t . On the lattice, a dynamical random environment is considered. It is described by the random field

$$\xi = \{ \xi_t(x) : x \in \mathbb{Z}^n, t \in \mathbb{Z}^+ \}$$

Note that the time is discrete. We assume that ξ is the result of independent copies of the same random variable taking values in some finite space \mathbb{S} . The space of configurations is given by $\tilde{\Omega} = \mathbb{S}^{\mathbb{Z}^n \times \mathbb{Z}^+}$. The values of the field for each $(t, x) \in \mathbb{Z}^{n+1}$, i.e. $\{ \xi_t(x) \}_{(x,t) \in \mathbb{Z}^{n+1}}$, are i.i.d random variables, distributed according to a given probability measure denoted by π .

The one step transition probability from position x at same time t to position y at the subsequent time step is given by

$$\mathbb{P} \{ X_{t+1} = y | X_t = x, \xi \} = P_0(y - x) + c(y - x, \xi_t(x))$$

where P_0 is the transition probability of a free random walk and c is the function which provides the influence of the environment on the particle's dynamic.

In order for the probability P to be well-defined, the following conditions must be fulfilled:

- $0 \leq P_0(u) + c(u, s) \leq 1 \quad \forall s \in \mathbb{S} \quad \forall u \in \mathbb{Z}^n;$
- $\sum_{u \in \mathbb{Z}^n} c(u, s) = 0 \quad \forall s \in \mathbb{S}.$

Moreover we assume that the random environment has the following property:

$$\sum_{s \in \mathbb{S}} c(u, s) \pi(s) = 0 \text{ for any } u \in \mathbb{Z}^n, \tag{1}$$

which means that P_0 is the mean transition probability.

Additionally, let P_0 and c be of bounded range. We denote by \mathbb{P}_ξ be the conditional probability with respect to the environment.

Let us define the ‘average’ transition probability

$$\bar{P}(u) = P_0(u) + \sum_{s \in \mathbb{S}} \pi(s) c(u, s). \tag{2}$$

We further assume that for some $b^c \in \mathbb{R}$,

$$\sum_{u \in \mathbb{Z}^n} uc(u, s) = b^c \quad \forall s \in \mathbb{S}. \quad (3)$$

Let $Y = \{Y_t\}_{t \in \mathbb{Z}^+}$ be the stochastic processes defined by $Y_t = X_t - tb$, where $b = b^0 + b^c$ and $b^0 = \sum_{u \in \mathbb{Z}^n} uP_0(u)$. Note that

$$\sum_{u \in \mathbb{Z}^n} (u - b) [P_0(u) + c(u, s)] = 0 \quad \forall s \in \mathbb{S}.$$

Theorem 2.1. *For almost every realisation ξ of the random environment we have*

$$\frac{1}{t}Y_t \Rightarrow \eta^2 U, \quad (4)$$

\mathbb{P}_ξ -a.s., where U is a standard normal vector and η^2 is the positive semidefinite matrix with entries

$$(\eta^2)_{ij} = \sum_{u \in \mathbb{Z}^n} (u_i - b_i)(u_j - b_j) \bar{P}(u). \quad (5)$$

3 Proofs

Lemma 3.1. *For every $\xi \in \tilde{\Omega}$, the process Y is a martingale under \mathbb{P}_ξ .*

Proof. This is a direct consequence of the definition of Y and (3).

Define $H_t = \mathbb{E}(Y_t Y_t')$, where Y_t' the transposed matrix, and the matrix $[Y]_t = ([Y^i, Y^j]_t)_{1 \leq i, j \leq n}$, and $H_t^\xi = \mathbb{E}_\xi(Y_t Y_t')$. Let also $K_t = \frac{1}{\sqrt{t}} I_n$, where I_n is the $n \times n$ identity matrix.

Lemma 3.2. *We have*

$$\begin{aligned} & \mathbb{E}_\xi \left[(Y_{r+1}^i - Y_r^i)(Y_{r+1}^j - Y_r^j) \right] \\ &= \sum_{y \in \mathcal{Y}} \mathbb{P}\{Y_r = y \mid \xi\} \sum_{u \in \mathbb{Z}^n} (u_i - b_i)(u_j - b_j) [P_0(u) + c(u, \xi_r(y))]. \end{aligned} \quad (6)$$

The above sum by y is taken over the countable set

$$\mathcal{Y} := \{z_1 + z_2 b \mid z_1, z_2 \in \mathbb{Z}\}.$$

(Note that $\mathbb{P}\{Y_t \in \mathcal{Y} \text{ for all } t \in \mathbb{N}\} = 1$).

Proof. By definition of Y and \mathbb{P}_ξ ,

$$\begin{aligned}
& \mathbb{E}_\xi \left[(Y_{r+1}^i - Y_r^i)(Y_{r+1}^j - Y_r^j) \right] \\
&= \mathbb{E} \left[(Y_{r+1}^i - Y_r^i)(Y_{r+1}^j - Y_r^j) \middle| \xi \right] = \mathbb{E} \left[\mathbb{E} \left\{ (Y_{r+1}^i - Y_r^i)(Y_{r+1}^j - Y_r^j) \middle| \xi, Y_r \right\} \middle| \xi \right] \\
&= \mathbb{E} \left[\sum_u (u_i - b_i - Y_r^i)(u_j - b_j - Y_r^j) [P_0(u - Y_r) + c(u - Y_r, \xi(Y_r))] \middle| \xi \right] \\
&= \mathbb{E} \left[\sum_u (u_i - b_i)(u_j - b_j) [P_0(u) + c(u, \xi(Y_r))] \middle| \xi \right] \\
&= \sum_{y \in \mathcal{Y}} \mathbb{P}\{Y_r = y \mid \xi\} \sum_{u \in \mathbb{Z}^n} (u_i - b_i)(u_j - b_j) [P_0(u) + c(u, \xi_r(y))].
\end{aligned}$$

□

Lemma 3.3. *We have*

$$\frac{1}{t}[M]_t \rightarrow \eta^2, \quad (7)$$

P_ξ -a.s. for a.a. ξ .

Proof. Note that for $1 \leq i, j \leq n$,

$$([Y]_t)_{ij} = \sum_{0 \leq r < t} \Delta_{r,ij},$$

where

$$\Delta_{r,ij} = [Y_{r+1}^i - Y_r^i][Y_{r+1}^j - Y_r^j].$$

Under \mathbb{P}_ξ a.s. on $\{Y_t = y\}$ the distribution of $Y_{t+1} - Y_t$ is $P_0(u) + c(u, \xi_t(y))$. Since under \mathbb{P}_ξ the random variables $Y_{t+1} - Y_t$ are independent of each other for different t , the statement of the lemma follows from the law of large numbers. □

Corollary 3.4. *Lemma 3.3 also holds P -a.s.*

Lemma 3.5. (i) *We have*

$$(H_{r+1})_{ij} - (H_r)_{ij} = \sum_{s \in \mathbb{S}} \pi(s) \sum_{u \in \mathbb{Z}^n} (u_i - b_i)(u_j - b_j) [P_0(u) + c(u, s)]. \quad (8)$$

(ii) We also have

$$\begin{aligned} & (H_{r+1}^\xi)_{ij} - (H_r^\xi)_{ij} \\ &= \sum_{y \in \mathcal{Y}} \mathbb{P}_\xi\{Y_r = y\} \sum_{u \in \mathbb{Z}^n} (u_i - b_i)(u_j - b_j) [P_0(u) + c(u, \xi_r(y))]. \end{aligned} \quad (9)$$

Proof (i) We start by noting that for $i, j \in \{1, \dots, n\}$,

$$\mathbb{E} \left((Y_{t+1}^i - Y_t^i) Y_t^j \right) = 0. \quad (10)$$

Indeed,

$$\begin{aligned} & \mathbb{E} \left((Y_{t+1}^i - Y_t^i) Y_t^j \right) = \mathbb{E} \mathbb{E} \left[(Y_{t+1}^i - Y_t^i) Y_t^j \middle| Y_t \right] \\ &= \sum_{y \in \mathcal{Y}} \mathbb{P}\{Y_t = y\} \sum_{u \in \mathbb{Z}^n} (y_i + u_i - b_i - y_i) y_j \bar{P}(u) = \sum_{y \in \mathcal{Y}} \mathbb{P}\{Y_t = y\} y_j \sum_{u \in \mathbb{Z}^n} (u_i - b_i) \bar{P}(u) = 0. \end{aligned}$$

By (10),

$$\begin{aligned} & (H_{r+1})_{ij} - (H_r)_{ij} = \mathbb{E} \left(Y_{t+1}^i Y_{t+1}^j - Y_t^i Y_t^j \right) \\ &= \mathbb{E} \left((Y_{t+1}^i - Y_t^i)(Y_{t+1}^j - Y_t^j) \right) + \mathbb{E} \left((Y_{t+1}^i - Y_t^i) Y_t^j \right) + \mathbb{E} \left(Y_t^i (Y_{t+1}^j - Y_t^j) \right) \\ &= \mathbb{E} \left((Y_{t+1}^i - Y_t^i)(Y_{t+1}^j - Y_t^j) \right). \end{aligned}$$

Conditioning on Y_t , we get

$$\begin{aligned} & (H_{r+1})_{ij} - (H_r)_{ij} = \sum_y P\{Y_t = y\} \sum_u (u_i - b_i)(u_j - b_j) [P_0(u) + c(u, \xi_r(y))] \\ &= \sum_u (u_i - b_i)(u_j - b_j) \bar{P}(u). \end{aligned}$$

(ii) (10) holds for \mathbb{E}_ξ too, since

$$\begin{aligned} & \mathbb{E} \mathbb{E} \left[(Y_{t+1}^i - Y_t^i) Y_t^j \middle| Y_t, \xi \right] \\ &= \sum_{y \in \mathcal{Y}} \mathbb{P}\{Y_t = y\} \sum_{u \in \mathbb{Z}^n} (y_i + u_i - b_i - y_i) y_j [P_0(u) + c(u, \xi_t(y))] \\ &= \sum_{y \in \mathcal{Y}} \mathbb{P}\{Y_t = y\} y_j \sum_{u \in \mathbb{Z}^n} (u_i - b_i) P_0(u) + \sum_{y \in \mathcal{Y}} \mathbb{P}\{Y_t = y\} y_j \sum_{u \in \mathbb{Z}^n} u_i c(u, \xi_t(y)) \\ &\quad - b_i \sum_{y \in \mathcal{Y}} \mathbb{P}\{Y_t = y\} y_j \sum_{u \in \mathbb{Z}^n} c(u, \xi_t(y)) \\ &= \sum_{y \in \mathcal{Y}} \mathbb{P}\{Y_t = y\} (-y_j b^c) + \sum_{y \in \mathcal{Y}} \mathbb{P}\{Y_t = y\} y_j b^c - 0 = 0. \end{aligned}$$

The proof continues as in (i). □

Lemma 3.6.

$$\frac{1}{t}H_t \rightarrow \eta^2, \quad \frac{1}{t}H_t^\xi \rightarrow \eta^2, \quad (11)$$

where η^2 is as in (5), \mathbb{P}_ξ -a.s for Π -a.a. ξ .

Proof. Let us only prove the second convergence in (11). By Lemma 3.5,

$$\begin{aligned} (H_t^\xi)_{ij} &= \sum_{r=0}^{t-1} \sum_{y \in \mathcal{Y}} \mathbb{P}_\xi\{Y_r = y\} \sum_{u \in \mathbb{Z}^n} (u_i - b_i)(u_j - b_j) [P_0(u) + c(u, \xi_r(y))] \\ &= t \sum_{u \in \mathbb{Z}^n} (u_i - b_i)(u_j - b_j) P_0(u) + \sum_{u \in \mathbb{Z}^n} (u_i - b_i)(u_j - b_j) \sum_{r=0}^{t-1} \sum_{y \in \mathcal{Y}} \mathbb{P}_\xi\{Y_r = y\} c(u, \xi_r(y)). \end{aligned}$$

The statement of the lemma would follow once we show that for every $s \in \mathbb{S}$ a.s.

$$\frac{\#\{(r, y) : r \leq t, Y_r = y, \xi_r(y) = s\}}{t} = \pi(s). \quad (12)$$

Since the events $\{Y_r = y\}$ and $\{\xi_r(y) = s\}$ are independent, so by the law of large numbers (12) holds \mathbb{P} -a.s. Hence (12) also holds \mathbb{P}_ξ -a.s for Π -a.a. ξ , otherwise, denoting the event of the left hand side of (12) by A , we would have

$$\mathbb{P}(A) = \int \mathbb{P}_\xi(A) \pi(d\xi) < 1.$$

□

Proof of Theorem 2.1 Theorem 2.1 by Küchler and Sørensen [KS99] and Lemmas 3.3 and 3.6 imply that P_ξ -a.s.

$$\frac{1}{t}Y_t \Rightarrow \eta^2 U, \quad (13)$$

where U a standard n -dimensional Gaussian vector.

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